COHEN-MACAULAY AUSLANDER ALGEBRAS OF GENTLE ALGEBRAS

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Abstract. For any gentle algebra $\Lambda = KQ/\langle I\rangle$, following Kalck, we describe the quiver and the relations for its Cohen-Macaulay Auslander algebra $\mathrm{Aus}(\mathrm{Gproj}\,\Lambda)$ explicitly, and obtain some properties, such as Λ is representation-finite if and only if $\mathrm{Aus}(\mathrm{Gproj}\,\Lambda)$ is; if Q has no loop and any indecomposable Λ-module is uniquely determined by its dimension vector, then any indecomposable $\mathrm{Aus}(\mathrm{Gproj}\,\Lambda)$-module is uniquely determined by its dimension vector.

1. Introduction

The concept of Gorenstein projective modules over any ring can be dated back to [4], where Auslander and Bridger introduced the modules of G-dimension zero over a Noetherian rings, and is formed by Enochs and Jenda [14]. This class of modules satisfies some good stable properties, becomes a main ingredient in the relative homological algebra, and is widely used in the representation theory of algebras and algebraic geometry, see e.g. [4, 6, 14, 10, 16, 8]. It also plays as an important tool to study the representation theory of Gorenstein algebra, see e.g. [6, 10, 16].

Gorenstein algebra Λ, where by definition Λ has finite injective dimension both as a left and a right Λ-module, is inspired from commutative ring theory. A fundamental result of Buchweitz [10] and Happel [16] states that for a Gorenstein algebra Λ, its singularity category is triangle equivalent to the stable category of Gorenstein projective (also called (maximal) Cohen-Macaulay) Λ-modules, which generalizes Rickard's result [22] on self-injective algebras.

For any Artin algebra Λ, denote by $\mathrm{Gproj}\,\Lambda$ its subcategory of Gorenstein projective modules. If $\mathrm{Gproj}\,\Lambda$ has only finitely many isomorphism classes of indecomposable objects, then Λ is called CM-*finite*. In this case, inspired by the definition of Auslander algebra, the Cohen-Macaulay Auslander algebra (also called the relative Auslander algebra) is defined to be $\mathrm{End}_\Lambda(\bigoplus_{i=1}^n E_i)^{op}$, where $E_1,\dots,E_n$ are all pairwise non-isomorphic indecomposable Gorenstein projective modules [7, 8, 19]. A CM-finite algebra Λ is Gorenstein if and only if $\mathrm{gl.dim}\,\mathrm{Aus}(\mathrm{Gproj}\,\Lambda) < \infty$ [19, 8]. Furthermore, for any two Gorenstein Artin algebras A and B which are CM-finite, if A and B are derived equivalent, then their Cohen-Macaulay Auslander algebras are also derived equivalent [21].

As an important class of Gorenstein algebras [15], gentle algebras were introduced in [3] as appropriate context for the investigation of algebras derived equivalent to hereditary algebras of type $\tilde{\mathbb{A}}_n$. Many important algebras are gentle, such as tilted algebras of type $\mathbb{A}_n$, algebras derived equivalent to $\mathbb{A}_n$-configurations of projective lines [11] and also the cluster-tilted algebras of type $\mathbb{A}_n$ [9], and type $\tilde{\mathbb{A}}_n$ [1]. It is interesting to notice that the class of gentle algebras is closed under derived equivalence [24]. Recently, Kalck [17] proves that the singularity category of an arbitrary gentle algebra is a finite product of n-cluster categories

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of type $\mathbb{A}_1$. From [17], it is easy to see that gentle algebras are CM-finite, which inspires us to study the properties of their Cohen-Macaulay Auslander algebras.

In this paper, our aim is to study the Cohen-Macaulay Auslander algebras of gentle algebras. Let $\Lambda = KQ/\langle I\rangle$ be a gentle algebra. First, we explicitly describe the quiver and relations of $\mathrm{Aus}(\mathrm{Gproj}\,\Lambda) = KQ^{Aus}/\langle I^{Aus}\rangle$, see Theorem 3.5. Second, we prove that Λ is representation-finite if and only if $\mathrm{Aus}(\mathrm{Gproj}\,\Lambda)$ is, see Theorem 4.4. Third, if Q has no loop, and any indecomposable Λ-module M is uniquely determined by its dimension vector, then any indecomposable $\mathrm{Aus}(\mathrm{Gproj}\,\Lambda)$-module N is uniquely determined by its dimension vector, see Theorem 4.6.

It is worth pointing out that in [13] we construct a desingularization of arbitrary quiver Grassmannians for finite-dimensional Gorenstein projective modules of 1-Gorenstein gentle algebras in terms of quiver Grassmannians for their Cohen-Macaulay Auslander algebras.

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2. Preliminaries

Throughout this paper, we always assume that K is an algebraically closed field. For any finite set S, we denote by $|S|$ the number of the elements in S. For a K-algebra, we always means a basic finite-dimensional associative K-algebra. For any algebra A, we denote by $\mathrm{gl.dim}\,A$ its *global dimension*. For an additive category $\mathcal{A}$, we denote by $\mathrm{ind}\,\mathcal{A}$ the isomorphism classes of indecomposable objects in $\mathcal{A}$.

Let $Q = (Q_0, Q_1)$ be a quiver (where Q_0 is the set of vertices and Q_1 is the set of arrows) and $\langle I\rangle$ an *admissible ideal* in the path algebra KQ which is generated by a set of relations I. Denote by (Q, I) the *associated bound quiver*. For any arrow α in Q we denote by $s(\alpha)$ its starting point and by $t(\alpha)$ its ending point. An *oriented path* (or path for short) of *length* $r \geq 1$ from a to b is a sequence $p = \alpha_1\alpha_2\dots\alpha_r$ of arrows α_i such that $t(\alpha_i) = s(\alpha_{i-1})$ for all $i = 2, \dots, r$, and $s(\alpha_r) = a$, $t(\alpha_1) = b$. A path of length $r \geq 1$ is called an *oriented cycle* whenever its source and target coincide. An oriented cycle of length 1 is called a *loop*.

2.1. Gentle algebras.

We first recall the definition of special biserial algebras and of gentle algebras.

Definition 2.1 ([25]). *The pair (Q, I) is called special biserial if it satisfies the following conditions.*

- *Each vertex of Q is the starting point of at most two arrows, and ending point of at most two arrows.*
- *For each arrow α in Q there is at most one arrow β such that $\alpha\beta \notin I$, and at most one arrow γ such that $\gamma\alpha \notin I$.*

Definition 2.2 ([3]). *The pair (Q, I) is called gentle if it is special biserial and moreover the following holds.*

- *The set I is generated by zero-relations of length 2.*
- *For each arrow α in Q there is at most one arrow β with $t(\beta) = s(\alpha)$ such that $\alpha\beta \in I$, and at most one arrow γ with $s(\gamma) = t(\alpha)$ such that $\gamma\alpha \in I$.*

A finite-dimensional algebra A is called *special biserial* (resp., *gentle*) if it has a presentation as $A = KQ/\langle I\rangle$ where (Q, I) is special biserial (resp., gentle).

Example 2.3. *(a) Let Q be the quiver as Figure 1 shows, and $I = \{\beta\alpha, \alpha\gamma_1, \gamma_1\beta\}$. Then $KQ/\langle I\rangle$ is a gentle algebra.*

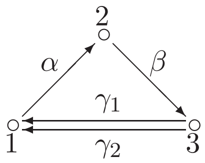

Figure 1. The quiver Q in Example 2.3 (a).

(b) Let Q be the quiver as Figure 2 shows, and $I = \{\alpha\beta, \beta\alpha, \gamma^2\}$. Then $\Lambda = KQ/\langle I\rangle$ is a gentle algebra.

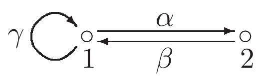

Figure 2. The quiver Q in Example 2.3 (b).

A classification of indecomposable modules over gentle algebras can be deduced from the work of Ringel [23] (see e.g. [12, 26]). For each arrow β, we denote by β^{-1} the formal inverse of β with $s(\beta^{-1}) = t(\beta)$ and $t(\beta^{-1}) = s(\beta)$. A word $w = c_1c_2\cdots c_n$ of arrows and their formal inverse is called a *string* of length $n \geq 1$ if $c_{i+1} \neq c_i^{-1}$, $s(c_i) = t(c_{i+1})$ for all $1 \leq i \leq n-1$, and no subword nor its inverse is in I. We define $(c_1c_2\cdots c_n)^{-1} = c_n^{-1}\cdots c_2^{-1}c_1^{-1}$, and $s(c_1c_2\cdots c_n) = s(c_n)$, $t(c_1c_2\cdots c_n) = t(c_1)$. We denote the length of w by $l(w)$. In addition, we also want to have strings of length 0; be definition, for any vertex $u \in Q_0$, there will be two strings of length 0, denoted by $1_{(u,1)}$ and $1_{(u,-1)}$, with both $s(1_{(u,i)}) = u = t(1_{(u,i)})$ for $i = -1, 1$, and we define $(1_{(u,i)})^{-1} = 1_{(u,-i)}$. We also denote by $\mathcal{S}(\Lambda)$ the set of all strings over $\Lambda = KQ/\langle I\rangle$.

Remark 2.4. *For any string $w \in \mathcal{S}(\Lambda)$, we have $w \neq w^{-1}$.*

Proof. If w is of length zero, then $w = 1_{(u,i)}$ for $i = 1$ or -1, and $w^{-1} = 1_{u,-i}$ which is different to w by the definition.

If $l(w) = n \geq 1$, then we assume that $w = c_1c_2\cdots c_n$. So $w^{-1} = c_n^{-1}\cdots c_2^{-1}c_1^{-1}$. Suppose for a contradiction that $w = w^{-1}$, which means $c_j = c_{n-j+1}^{-1}$ for $j = 1, \dots, n$. If $n = 2k$ for some integer k, then $c_k = c_{k+1}^{-1}$, a contradiction to the definition of strings. If $n = 2k + 1$ for some integer k, then $c_{k+1} = c_{k+1}^{-1}$, which yields a contradiction. So $w \neq w^{-1}$. $\square$

A *band* $b = \alpha_1\alpha_2\cdots\alpha_{n-1}\alpha_n$ is defined to be a string b with $t(\alpha_1) = s(\alpha_n)$ such that each power b^m is a string, but b itself is not a proper power of any strings. We denote by $\mathcal{B}(\Lambda)$ the set of all bands over Λ.

On $\mathcal{S}(\Lambda)$, we consider the equivalence relation ρ which identifies every string C with its inverse C^{-1}. On $\mathcal{B}(\Lambda)$, we consider the equivalence relation ρ' which identifies every string $C = c_1 \dots c_n$ with the cyclically permuted strings $C_{(i)} = c_ic_{i+1}\cdots c_nc_1\cdots c_{i-1}$ and their inverses $C_{(i)}^{-1}$, $1 \leq i \leq n$. We choose a complete set $\underline{\mathcal{S}}(\Lambda)$ of representatives of $\mathcal{S}(\Lambda)$ relative to ρ, and a complete set $\underline{\mathcal{B}}(\Lambda)$ of representatives of $\mathcal{B}(\Lambda)$ relative to ρ'.

Butler and Ringel showed that each string w defines a unique string module $M(w)$, each band b yields a family of band modules $M(b,m,\phi)$ with $m \geq 1$ and $\phi \in \mathrm{Aut}(K^m)$. Equivalently, one can consider certain quiver morphism $\sigma : S \to Q$ (for strings) and $\beta : B \to Q$ (for bands), where S and B are of types $\mathbb{A}_n$ and $\tilde{\mathbb{A}}_n$, respectively. Then string and band modules are given as pushforwards $\sigma_*(M)$ and $\beta_*(R)$ of indecomposable KS-modules M and indecomposable regular KB-modules R, respectively (see e.g. [26]). Let $\underline{\mathrm{Aut}}(K^m)$ be a complete set of representatives of indecomposable automorphisms of K-spaces with respect to similarity.

Theorem 2.5 ([12]). *The modules $M(w)$ with $w \in \underline{\mathcal{S}}(\Lambda)$, and the modules $M(b,m,\phi)$ with $b \in \underline{\mathcal{B}}(\Lambda)$, $m \geq 1$ and $\phi \in \underline{\mathrm{Aut}}(K^m)$, provide a complete list of indecomposable (and pairwise non-isomorphic) Λ-modules.*

In practice, a string w is of form $\alpha_1^{\epsilon_1}\alpha_2^{\epsilon_2}\cdots\alpha_n^{\epsilon_n}$ for $\alpha_i \in Q_1$ and $\epsilon_i = \pm 1$ for all $1 \leq i \leq n$. So w can be viewed as a walk in Q:

$$w: \quad b_1 \overset{\alpha_1}{\text{——}} b_2 \overset{\alpha_2}{\text{——}} \cdots \overset{\alpha_{n-1}}{\text{——}} b_n \overset{\alpha_n}{\text{——}} b_{n+1},$$

where $b_1, b_2 \ldots, b_{n+1}$ are vertices of Q and α_i is an arrow from b_{i+1} to b_i if $\epsilon_i = 1$, or an arrow from b_i to b_{i+1} if $\epsilon_i = -1$, for each $1 \leq i \leq n$. In this way, the equivalence relation ρ induces that

$$w: \quad b_1 \overset{\alpha_1}{\text{——}} b_2 \overset{\alpha_2}{\text{——}} \cdots \overset{\alpha_{n-1}}{\text{——}} b_n \overset{\alpha_n}{\text{——}} b_{n+1},$$

is equivalent to

$$w^{-1}: \quad b_{n+1} \overset{\alpha_n}{\text{——}} b_n \overset{\alpha_{n-1}}{\text{——}} \cdots \overset{\alpha_2}{\text{——}} b_2 \overset{\alpha_1}{\text{——}} b_1.$$

It is similar to interpret ρ' if w is a band. We denote by $v \sim w$ for any two strings v, w if v is equivalent to w under ρ.

For any string $w = c_1 \ldots c_n$, or $w = 1_{(u,j)}$, let $u_w(i) = t(c_{i+1})$, $0 \leq i < n$, and $u_w(n) = s(w) = s(c_n)$. Given a vertex $v \in Q_0$, let $I_w(v) = \{i | u_w(i) = v\} \subseteq \{0, 1, \ldots, n\}$. Denote by $k_w(v) = |I_w(v)|$. We associate a vector $(k_w(v))_{v \in Q_0}$ to the string w, which is denoted by $\underline{\dim}\, w$, and call it the *dimension vector* of w. From [12], we get that $\underline{\dim}\, w = \underline{\dim}\, M(w)$.

Note that if a gentle algebra Λ is representation-finite, then there is no band module in $\mathrm{mod}\,\Lambda$, and so all the indecomposable modules over Λ are string modules.

2.2. **Singularity categories and Gorenstein algebras.** Let Λ be a finite-dimensional K-algebra. Let $\mathrm{mod}\,\Lambda$ be the category of finitely generated left Λ-modules, and $\mathrm{proj}\,\Lambda$ the subcategory of finitely generated projective Λ-modules. For an arbitrary Λ-module ${}_\Lambda X$, we denote by $\mathrm{proj.dim}_\Lambda X$ (resp. $\mathrm{inj.dim}_\Lambda X$) the projective dimension (resp. the injective dimension) of the module ${}_\Lambda X$. A Λ-module G is *Gorenstein projective*, if there is an exact sequence

$$P^\bullet : \cdots \to P^{-1} \to P^0 \xrightarrow{d^0} P^1 \to \cdots$$

of projective Λ-modules, which stays exact under $\mathrm{Hom}_\Lambda(-,\Lambda)$, and such that $G \cong \mathrm{Ker}\, d^0$. We denote by $\mathrm{Gproj}(\Lambda)$ the subcategory of Gorenstein projective Λ-modules.

Definition 2.6 ([5, 6, 16]). *A finite-dimensional algebra Λ is called a Gorenstein (or Iwanaga-Gorenstein) algebra if Λ satisfies* $\mathrm{inj.dim}\,\Lambda_\Lambda < \infty$ *and* $\mathrm{inj.dim}_\Lambda \Lambda < \infty$.

Observe that for a Gorenstein algebra Λ, we have $\mathrm{inj.dim}_\Lambda \Lambda = \mathrm{inj.dim}\,\Lambda_\Lambda$, see e.g. [16, Lemma 6.9]; the common value is denoted by $\mathrm{G.dim}\,\Lambda$. If $\mathrm{G.dim}\,\Lambda \leq d$, we say that Λ is *d-Gorenstein*.

For an algebra Λ, the *singularity category* of Λ is defined to be the quotient category $D^b_{sg}(\Lambda) := D^b(\Lambda)/K^b(\mathrm{proj}\,\Lambda)$ [10, 16, 20]. Note that $D^b_{sg}(\Lambda)$ is zero if and only if $\mathrm{gl.dim}\,\Lambda < \infty$ [16].

Theorem 2.7 ([10, 16]). *Let Λ be a finite-dimensional algebra. Then* $\mathrm{Gproj}(\Lambda)$ *is a Frobenius category with the projective modules as the projective-injective objects. If Λ is Gorenstein, then the stable category* $\underline{\mathrm{Gproj}}(\Lambda)$ *is triangle equivalent to the singularity category $D^b_{sg}(\Lambda)$ of Λ.*

An algebra is of *finite Cohen-Macaulay type*, or simply, *CM-finite*, if there are only finitely many isomorphism classes of indecomposable finitely generated Gorenstein projecitve modules. Clearly, Λ is CM-finite if and only if there is a finitely generated module E such that $\mathrm{Gproj}\,\Lambda = \mathrm{add}\,E$. In this way, E is called to be a *Gorenstein projective generator*. If the global dimension of Λ is finite, then $\mathrm{Gproj}\,\Lambda = \mathrm{proj}\,\Lambda$, which implies that Λ is CM-finite. If Λ is self-injective, then $\mathrm{Gproj}\,\Lambda = \mathrm{mod}\,\Lambda$, so Λ is CM-finite if and only if Λ is representation-finite.

Let Λ be a CM-finite algebra, $E_1, \dots, E_n$ all the pairwise non-isomorphic indecomposable Gorenstein projective Λ-modules. Put $E = \oplus_{i=1}^n E_i$. Then E is a Gorenstein projective generator. We call $\mathrm{Aus}(\mathrm{Gproj}\,\Lambda) := (\mathrm{End}_\Lambda E)^{op}$ the *Cohen-Macaulay Auslander algebra* (also called *relative Auslander algebra*) of Λ.

Geiß and Reiten [15] prove that gentle algebras are Gorenstein algebras, so their Cohen-Macaulay Auslander algebras have finite global dimensions [19]. The singularity category of a gentle algebra is characterized by Kalck in [17], we recall it as follows. For a gentle algebra $\Lambda = KQ/\langle I\rangle$, we denote by $\mathcal{C}(\Lambda)$ the set of equivalence classes (with respect to cyclic permutation) of *repetition-free* cyclic paths $\alpha_1 \dots \alpha_n$ in Q such that $\alpha_i\alpha_{i+1} \in I$ for all i, where we set $n+1 = 1$. Moreover, we set $l(c)$ to be the *length* of the cycle $c \in \mathcal{C}(\Lambda)$, i.e. $l(\alpha_1 \dots \alpha_n) = n$.

For every arrow $\alpha \in Q_1$, there is at most one cycle $c \in \mathcal{C}(\Lambda)$ containing α. In fact, if there are two different elements $c, c' \in \mathcal{C}(\Lambda)$ such that α lies on both of them, then the definition of $\mathcal{C}(\Lambda)$ implies that there exist arrows β, γ_1 and γ_2 such that $\gamma_1 \neq \gamma_2$, $s(\gamma_1) = t(\beta) = s(\gamma_2)$ and $\gamma_1\beta, \gamma_2\beta \in I$, a contradiction to that Λ is gentle. We define $R(\alpha)$ to be the *left ideal* $\Lambda\alpha$ generated by α. It follows from the definition of gentle algebras that this is a direct summand of the radical $\mathrm{rad}\,P_{s(\alpha)}$ of the indecomposable projective Λ-module $P_{s(\alpha)} = \Lambda e_{s(\alpha)}$, where $e_{s(\alpha)}$ is the idempotent corresponding to $s(\alpha)$. In fact, all radical summands of indecomposable projective modules arise in this way, see e.g. [17].

Theorem 2.8 ([17]). *Let $\Lambda = KQ/\langle I\rangle$ be a gentle algebra. Then*
(i) $\mathrm{ind}\,\mathrm{Gproj}(\Lambda) = \mathrm{ind}\,\mathrm{proj}\,\Lambda \bigcup \{R(\alpha_1), \dots, R(\alpha_n) | c = \alpha_1 \cdots \alpha_n \in \mathcal{C}(\Lambda)\}$.
(ii) *There is an equivalence of triangulated categories*

$$D^b_{sg}(\Lambda) \simeq \prod_{c \in \mathcal{C}(\Lambda)} \frac{D^b(K)}{[l(c)]},$$

where $D^b(K)/[l(c)]$ denotes the triangulated orbit category, see [18].

From Theorem 2.8 or its proof in [17], we get that $\underline{\mathrm{Gproj}}(\Lambda) \simeq D^b_{sg}(\Lambda)$ is equivalent to a semisimple abelian category and therefore itself is semisimple abelian. In particular, $\underline{\mathrm{Hom}}_\Lambda(R(\alpha), R(\alpha')) \cong \delta_{\alpha\alpha'}K$ for any two non-projective indecomposable Gorenstein projective modules $R(\alpha), R(\alpha')$.

3. Cohen-Macaulay Auslander algebras of gentle algebras

Let $\Lambda = KQ/\langle I\rangle$ be a gentle algebra. It is easy to get the following lemma.

Lemma 3.1. *Let $\Lambda = KQ/\langle I\rangle$ be a gentle algebra. Then Λ is CM-finite.*

Proof. From Theorem 2.8, we get that

$$\operatorname{ind}\operatorname{Gproj}(\Lambda) = \operatorname{ind}\operatorname{proj}\Lambda \bigcup \{R(\alpha_1), \dots, R(\alpha_n) | c = \alpha_1 \cdots \alpha_n \in \mathcal{C}(\Lambda)\}.$$

So every non-projective indecomposable Gorenstein projective Λ-module is of form $R(\alpha)$ for some arrow α. Furthermore, there are only finitely many arrows, and then Λ is CM-finite. $\square$

From Λ, we construct a bound quiver (Q^{Aus}, I^{Aus}) as follows:

• the set of vertices $Q_0^{Aus} := Q_0 \bigsqcup Q_1^{cyc}$, where $Q_1^{cyc} = \{\alpha | \exists\, c \in \mathcal{C}(\Lambda)$ such that α lies on $c\}$;

• the set of arrows $Q_1^{Aus} := Q_1^{ncyc} \bigsqcup (Q_1^{cyc})^{\pm}$, where $Q_1^{ncyc} = Q_1 \setminus Q_1^{cyc}$ (i.e. arrows do not lie on any cyclic paths in $\mathcal{C}(\Lambda)$), $(Q_1^{cyc})^+ = \{\alpha^+ : s(\alpha) \to \alpha | \alpha \in Q_1^{cyc}\}$ and $(Q_1^{cyc})^- = \{\alpha^- : \alpha \to t(\alpha) | \alpha \in Q_1^{cyc}\}$.

• the set of relations $I^{Aus} := \{\beta^+\alpha^- | \beta\alpha \in I$ with $\alpha, \beta \in Q_1^{cyc}\} \bigcup \{\beta\alpha | \beta\alpha \in I$ with $\alpha, \beta \in Q_1^{ncyc}\}$.

Note that if $\mathcal{C}(\Lambda) = \emptyset$, then $(Q^{Aus}, I^{Aus}) = (Q, I)$.

In this section, we prove that $KQ^{Aus}/\langle I^{Aus}\rangle$ is isomorphic to the Cohen-Macaulay Auslander algebra of the gentle algebra $\Lambda = KQ/\langle I\rangle$.

Example 3.2. *(a) Keep the notations as in Example 2.3 (a). Then the quiver Q^{Aus} of the gentle algebra $KQ/\langle I\rangle$ is as Figure 3 shows, and $I^{Aus} = \{\alpha^+\gamma_1^-, \beta^+\alpha^-, \gamma_1^+\beta^-\}$.*

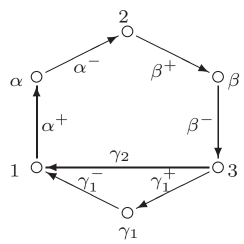

Figure 3. The quiver Q^{Aus} of $KQ/\langle I\rangle$ for Example 2.3 (a).

(b) Keep the notations as in Example 2.3 (b). Then the quiver Q^{Aus} of the gentle algebra $KQ/\langle I\rangle$ is as Figure 4 shows, and $I^{Aus} = \{\gamma^+\gamma^-, \alpha^+\beta^-, \beta^+\alpha^-\}$.

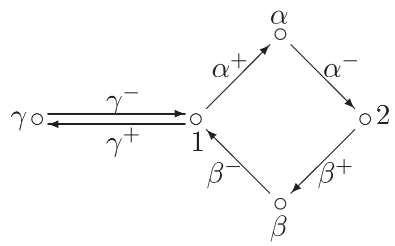

Figure 4. The quiver Q^{Aus} of $KQ/\langle I\rangle$ for Example 2.3 (b).

For any two Λ-modules M, N and any subcategory $\mathcal{D}$ of $\operatorname{mod}\Lambda$ containing M, N, we denote by $\operatorname{irr}_{\mathcal{D}}(M, N)$ the space of irreducible morphisms from M to N in $\mathcal{D}$.

From Theorem 2.8, we get that

$$\operatorname{ind}\operatorname{Gproj}(\Lambda) = \operatorname{ind}\operatorname{proj}\Lambda \bigcup \{R(\alpha_1), \dots, R(\alpha_n) | c = \alpha_1 \cdots \alpha_n \in \mathcal{C}(\Lambda)\}.$$

Furthermore, let $c \in \mathcal{C}(\Lambda)$ be a cycle, which we label as follows: $1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} n \xrightarrow{\alpha_n} 1$. Then from the proof of [17, Theorem 2.5], there are short exact sequences

$$0 \to R(\alpha_i) \xrightarrow{a_i} P_i \xrightarrow{b_i} R(\alpha_{i-1}) \to 0, \tag{1}$$

for all $i = 1, \dots, n$, where we set $\alpha_0 = \alpha_n$.

Lemma 3.3. *Keep the notations as above. Then a_i, b_i in sequence (1) are irreducible morphisms in* $\operatorname{Gproj}\Lambda$ *for all $i = 1, \ldots, n$. Furthermore,*

(i)
$$\dim_K \operatorname{irr}_{\operatorname{Gproj}\Lambda}(R(\alpha_i), P_i)) = 1 \text{ and } \dim_K \operatorname{irr}_{\operatorname{Gproj}\Lambda}(P_i, R(\alpha_{i-1})) = 1,$$
for all $i = 1, \ldots, n$.

(ii) *For any indecomposable projective module P not isomorphic to P_i, we have*
$$\operatorname{irr}_{\operatorname{Gproj}\Lambda}(R(\alpha_i), P)) = 0 \text{ and } \operatorname{irr}_{\operatorname{Gproj}\Lambda}(P, R(\alpha_{i-1})) = 0,$$
for all $i = 1, \ldots, n$.

(iii) *For any two non-projective indecomposable Gorenstein projective modules $R(\alpha)$ and $R(\alpha')$, we have* $\operatorname{irr}_{\operatorname{Gproj}\Lambda}(R(\alpha), R(\alpha')) = 0$.

Proof. Note that $R(\alpha_i)$ is indecomposable and sequence (1) is not split for any $\alpha_n \ldots \alpha_1 \in \mathcal{C}(\Lambda)$ and each $i = 1, \ldots, n$. We need to check that sequence (1) is an almost split sequence in $\operatorname{Gproj}\Lambda$ for each $i = 1, \ldots, n$.

For any Gorenstein projective module M, and a morphism $v : M \to R(\alpha_{i-1})$ which is not a retraction, since $\underline{\operatorname{Gproj}}(\Lambda)$ is a semisimple category, and $R(\alpha_{i-1})$ is a simple object in $\underline{\operatorname{Gproj}}(\Lambda)$, we get that $v = 0$ in $\underline{\operatorname{Gproj}}(\Lambda)$. So v factors through a projective module P as $v = v_2 v_1$ for some morphisms $v_1 : M \to P$ and $v_2 : P \to R(\alpha_{i-1})$. It is easy to see that v_2 factors through b_i as $v_2 = b_i v_3$ for some morphism $v_3 : P \to P_i$, which implies $v = v_2 v_1 = b_i v_3 v_1$, so b_i is right almost split and then sequence (1) is almost split.

$$\begin{tikzcd}
R(\alpha_i) \arrow[r, "a_i"] & P_i \arrow[r, "b_i"] & R(\alpha_{i-1}) \\
& P \arrow[u, dotted, "v_3"] \arrow[ur, dotted, "v_2"] & M. \arrow[u, "v"'] \arrow[l, dotted, "v_1"]
\end{tikzcd}$$

(i) For any other irreducible morphism $a_i' : R(\alpha_i) \to P_i$, since $\operatorname{Ext}^1_\Lambda(R(\alpha_{i-1}), P_i) = 0$, there exists a morphism $f : P_i \to P_i$ such that $a_i' = f a_i$. Note that a_i is not a section, so f is a retraction and then an isomorphism, so $\dim_K \operatorname{irr}_{\operatorname{Gproj}\Lambda}(R(\alpha_i), P_i)) = 1$.

It is similar to prove that $\dim_K \operatorname{irr}_{\operatorname{Gproj}\Lambda}(P_i, R(\alpha_{i-1})) = 1$, we omit the proof here.

(ii) follows from that sequence (1) is almost split.

(iii) If $\alpha \neq \alpha'$, then $\underline{\operatorname{Hom}}_\Lambda(R(\alpha), R(\alpha')) = 0$, so $\operatorname{irr}_{\operatorname{Gproj}\Lambda}(R(\alpha), R(\alpha')) = 0$. If $\alpha = \alpha'$, then by the proof of Theorem 2.8 in [17], we get that $\operatorname{End}_\Lambda(R(\alpha)) = K$. So $\operatorname{irr}_{\operatorname{Gproj}\Lambda}(R(\alpha), R(\alpha)) = 0$. □

Since $\operatorname{proj}\Lambda \subset \operatorname{Gproj}\Lambda$, for any indecomposable projective Λ-modules P_1, P_2, we get that $\operatorname{irr}_{\operatorname{Gproj}\Lambda}(P_1, P_2) \subseteq \operatorname{irr}_{\operatorname{proj}\Lambda}(P_1, P_2)$.

Lemma 3.4. *Let $\Lambda = KQ/\langle I \rangle$ be a gentle algebra. Let P_1, P_2 be two indecomposable projective Λ-modules with their corresponding vertices v_1, v_2 respectively. For any irreducible morphism $f : P_1 \to P_2$ in* $\operatorname{proj}\Lambda$ *which is induced by an arrow $\alpha : v_2 \to v_1$, then*

(i) *if α lies on a cycle in $\mathcal{C}(\Lambda)$, then f is not irreducible in* $\operatorname{Gproj}\Lambda$, *in particular, f factors through $R(\alpha)$ as a composition of two irreducible morphisms in* $\operatorname{Gproj}\Lambda$.

(ii) *if α does not lie on any cycle in $\mathcal{C}(\Lambda)$, then f is irreducible in* $\operatorname{Gproj}\Lambda$.

Proof. (i) If α lies on a cycle $c \in \mathcal{C}(\Lambda)$, we assume that c is of form $\cdots v_3 \xrightarrow{\gamma} v_2 \xrightarrow{\alpha} v_1 \xrightarrow{\beta} 0 \cdots$ (where the vertices can be coincided), then there exist two short exact sequences
$$0 \to R(\alpha) \xrightarrow{a_1} P_2 \xrightarrow{b_1} R(\gamma) \to 0 \text{ and } 0 \to R(\beta) \xrightarrow{a_2} P_1 \xrightarrow{b_2} R(\alpha) \to 0,$$

with $a_1b_2 = f$. So f is not irreducible in $\operatorname{Gproj}\Lambda$. Lemma 3.3 yields that a_1, b_2 are irreducible in $\operatorname{Gproj}\Lambda$, and then (i) follows.

(ii) Since $f \in \operatorname{irr}_{\operatorname{proj}\Lambda}(P_1, P_2)$, we get that f is neither a section nor a retraction. Suppose for a contradiction that f factors through a module $M \in \operatorname{Gproj}\Lambda$ as $f = f_2f_1$ for some morphisms $f_1 : P_1 \to M$ and $f_2 : M \to P_2$, with neither f_1 a section nor f_2 a retraction. Then $M \notin \operatorname{proj}\Lambda$, so $M = M_1 \oplus M_2$ with M_1 projective and the indecomposable direct summands of M_2 non-projective. Note that $M_2 \neq 0$. For any non-projective indecomposable Gorenstein projective module R_i, there exist indecomposable projective modules P_i, P_{i+1} and non-projective Gorenstein projective modules R_{i-1}, R_{i+1} such that the following sequences are exact

$$0 \to R_i \to P_i \to R_{i-1} \to 0, \quad 0 \to R_{i+1} \to P_{i+1} \to R_i \to 0. \tag{2}$$

So by doing direct sum of the exact sequences as in sequence (2) for all indecomposable direct summands of M_2, there exist two exact sequences

$$0 \to N_1 \xrightarrow{a_1} P_{M_2} \xrightarrow{b_1} M_2 \to 0, \quad 0 \to M_2 \xrightarrow{a_2} Q_{M_2} \xrightarrow{b_2} N_2 \to 0, \tag{3}$$

where P_{M_2}, Q_{M_2} are projective with their indecomposable direct summands corresponding to vertices lying on cycles in $\mathcal{C}(\Lambda)$, and N_1, N_2 are Gorenstein projective modules with their indecomposable direct summands non-projective. Then for M, there exist two exact sequences

$$0 \to N_1 \xrightarrow{c_1} M_1 \oplus P_{M_2} \xrightarrow{d_1} M \to 0, \quad 0 \to M \xrightarrow{c_2} M_1 \oplus Q_{M_2} \xrightarrow{d_2} N_2 \to 0. \tag{4}$$

The proof can be broken into the following two cases.

Case (1). The vertex v_1 does not lie on any cycle in $\mathcal{C}(\Lambda)$. Then f_1 factors through d_1 as the following diagram shows:

$$\begin{tikzcd}
 & P_1 \arrow[r, "f"] \arrow[d, dotted, "f_1'"] \arrow[dr, "f_1"] & P_2 \\
N_1 \arrow[r, "c_1"] & M_1 \oplus P_{M_2} \arrow[r, "d_1"] & M. \arrow[u, "f_2"']
\end{tikzcd}$$

So $f = f_2d_1f_1'$. If f_1' is not a section, then f_2d_1 is a retraction since f is irreducible in $\operatorname{proj}\Lambda$ and $M_1 \oplus P_{M_2}$ is projective, which yields that f_2 is a retraction, giving a contradiction. So f_1' is a section, which implies that P_1 is a direct summand of M_1 by the assumption that the vertex 1 does not lie on any cycle in $\mathcal{C}(\Lambda)$. Since M_1 is a direct summand of M, we get that P_1 is a direct summand of M, i.e. f_1 is a section, giving a contradiction.

Case (2). The vertex v_1 lies on some cycle in $\mathcal{C}(\Lambda)$. Then there is a cycle $c \in \mathcal{C}(\Lambda)$ such that v_1 lies on c. So we assume that c locally is $\cdots \xrightarrow{\alpha_1} v_3 \xrightarrow{\alpha_2} v_1 \xrightarrow{\alpha_3} \cdots$. Let P_3 be the indecomposable projective module corresponding to the vertex v_3. Then there are two exact sequences:

$$0 \to R(\alpha_2) \xrightarrow{u_1} P_3 \xrightarrow{v_1} R(\alpha_1) \to 0, \quad 0 \to R(\alpha_3) \xrightarrow{u_2} P_1 \xrightarrow{v_2} R(\alpha_2) \to 0. \tag{5}$$

Similar to Case (1), we get that f_1 factors through d_1 as the following diagram shows:

$$\begin{tikzcd}
 & P_1 \arrow[r, "f"] \arrow[d, dotted, "f_1'"] \arrow[dr, "f_1"] & P_2 \\
N_1 \arrow[r, "c_1"] & M_1 \oplus P_{M_2} \arrow[r, "d_1"] & M. \arrow[u, "f_2"']
\end{tikzcd}$$

Then $f = f_2 d_1 f_1'$. If f_1' is not a section, then $f_2 d_1$ is a retraction since f is irreducible in $\operatorname{proj}\Lambda$ and $M_1 \oplus P_{M_2}$ is projective, which yields that f_2 is a retraction, giving a contradiction. So f_1' is a section.

If f_1' induces that P_1 is a direct summand of M_1, and then it is a direct summand of M. By our construction, we get that $f_1 : P_1 \to M$ is a section, giving a contradiction. So f_1' induces that P_1 is a direct summand of P_{M_2}. By our construction, we know that $R(\alpha_2)$ is a direct summand of M_2. So f factors through $v_2 : P_1 \to R(\alpha_2)$ as $f = g_2 v_2$ for some morphism $g_2 : R(\alpha_2) \to P_2$. From sequence (5), we get that g_2 factors through u_1 as $g_2 = g_2' u_1$ for some morphism $f_2' : P_3 \to P_2$ since $\operatorname{Ext}^1_\Lambda(R(\alpha_1), P_2) = 0$. Then $f = f_2' u_1 v_2$. Since $u_1 v_2 : P_1 \to P_3$ is the morphism induced by the arrow α_2, it is not a section. Therefore, f_2' is a retraction and then an isomorphism. So f is the morphism induced by the arrow α_2. However, f is the morphism induced by the arrow α, so $\alpha_2 = \alpha$. Recall that α does not lie on any cycle in $\mathcal{C}(\Lambda)$, giving a contradiction.

To sum up, f is an irreducible morphism in $\operatorname{Gproj}(\Lambda)$. □

Theorem 3.5. *Let $\Lambda = KQ/\langle I \rangle$ be a gentle algebra. Then the Cohen-Macaulay Auslander algebra of Λ is isomorphic to $KQ^{Aus}/\langle I^{Aus} \rangle$.*

Proof. Note that

$$\operatorname{ind}\operatorname{Gproj}(\Lambda) = \operatorname{ind}\operatorname{proj}\Lambda \bigcup \{R(\alpha_1), \dots, R(\alpha_n) | c = \alpha_1 \cdots \alpha_n \in \mathcal{C}(\Lambda)\}.$$

Lemma 3.3 and Lemma 3.4 characterize all the irreducible morphisms in $\operatorname{Gproj}\Lambda$, from them, it is easy to see that Q^{Aus} is the quiver of the Cohen-Macaulay Auslander algebra of Λ. In fact, the vertex $i \in Q_0 \subseteq Q_0^{Aus}$ corresponds to the corresponding indecomposable projective Λ-module P_i; the vertex $\alpha \in Q_1^{cyc} \subseteq Q_0^{Aus}$ corresponding to the Λ-module $R(\alpha)$; the arrow $\beta \in Q_1^{ncyc} \subseteq Q_1^{Aus}$ corresponds to the irreducible morphism $P_{t(\beta)} \to P_{s(\beta)}$ induced by $\beta \in Q_1$, see Lemma 3.4 (ii). The arrow α^- (resp. α^+) corresponds to the irreducible morphism $P_{t(\alpha)} \xrightarrow{b} R(\alpha)$ (resp. $R(\alpha) \xrightarrow{a} P_{s(\alpha)}$), see Lemma 3.3 and Lemma 3.4 (i). Note that b is surjective and a is injective.

So $\operatorname{Aus}(\operatorname{Gproj}\Lambda)$ is isomorphic to $KQ^{Aus}/\langle I^A \rangle$ for some admissible ideal $\langle I^A \rangle$. Recall that

$$I^{Aus} = \{\beta^+\alpha^- | \beta\alpha \in I \text{ with } \alpha, \beta \in Q_1^{cyc}\} \bigcup \{\beta\alpha | \beta\alpha \in I \text{ with } \alpha, \beta \in Q_1^{ncyc}\}.$$

From the above, it is easy to see that $\langle I^{Aus} \rangle \subseteq \langle I^A \rangle$. Assume that $l = \sum_{i=1}^t k_i l_i \in I^A$, where $l_1, \dots, l_t$ are paths in KQ^{Aus} and $k_i \neq 0$ for $1 \leq i \leq t$. We can also assume that the starting points and the ending points of all the $l_i, 1 \leq i \leq t$ are same, which are denoted by $s(l)$, $t(l)$ respectively. The proof can be broken into the following four cases.

Case (1). $s(l), t(l) \in Q_0 \subseteq Q_0^{Aus}$. We can view l to be an element in KQ after replacing all the subpaths $\alpha^-\alpha^+$ by α, and denote it by $\pi(l)$. Let us view the arrows as irreducible morphisms. For any arrow $\alpha \in Q_1^{cyc}$, the irreducible morphism from $P_{t(\alpha)}$ to $P_{s(\alpha)}$ in $\operatorname{proj}\Lambda$ induced by α is equal to the combination of the irreducible morphisms in $\operatorname{Gproj}\Lambda$ induced by the arrows α^- and α^+. So the morphism from $P_{t(l)}$ to $P_{s(l)}$ induced by $\pi(l)$ in $\operatorname{proj}\Lambda$ is equal to the one induced by l in $\operatorname{Gproj}\Lambda$. Since $l \in I^A$, the morphism from $P_{t(l)}$ to $P_{s(l)}$ induced by l is zero, and then the morphism from $P_{t(l)}$ to $P_{s(l)}$ induced by $\pi(l)$ is also zero. So $\pi(l) \in \langle I \rangle$, and then $\pi(l_i) \in \langle I \rangle$ for any $1 \leq i \leq t$, since $\langle I \rangle$ is generated by zero-relations of length two. In other words, for each $1 \leq i \leq t$, there exist two arrows α, β in Q such that $\beta\alpha \in I$ and $\beta\alpha$ is a subpath of $\pi(l_i)$. If $\alpha \in Q_1^{ncyc}$, then $\beta \in Q_1^{ncyc}$, and so $\beta\alpha \in I^{Aus}$, which implies that $l_i \in \langle I^{Aus} \rangle$; if $\alpha \in Q_1^{cyc}$, then $\beta \in Q_1^{cyc}$ and so $\beta^+\alpha^- \in I^{Aus}$. It is easy to see that $\beta^+\alpha^-$ is a subpath of l_i, which implies that $l_i \in \langle I^{Aus} \rangle$. Therefore, we have $l_i \in \langle I^{Aus} \rangle$ for each i, and then $l \in \langle I^{Aus} \rangle$.

Case (2). $s(l) = \alpha \in Q_1^{cyc} \subseteq Q_0^{Aus}, t(l) \in Q_0 \subseteq Q_0^{Aus}$. Since there is only one arrow α^- starting from α, we can assume $l = l'\alpha^-$ where l' is some element in KQ^{Aus} starting from $t(\alpha)$. Viewing the arrows as irreducible morphisms, since α^+ corresponds to an injective morphism, we get that $l = l'\alpha^- \in \langle I^A \rangle$ if and only if $l\alpha^+ \in \langle I^A \rangle$. Then $l\alpha^+$ satisfies Case (1), which implies that it is in $\langle I^{Aus} \rangle$. Since $\langle I^{Aus} \rangle$ is generated by zero-relations of length two and $\alpha^-\alpha^+ \notin \langle I^{Aus} \rangle$, we get that $l \in \langle I^{Aus} \rangle$.

Case (3). $s(l) \in Q_0 \subseteq Q_0^{Aus}, t(l) = \alpha \in Q_1^{cyc} \subseteq Q_0^{Aus}$. It is similar to Case (2), only need note that α^- corresponds to a surjective morphism.

Case (4). $s(l) = \alpha, t(l) = \beta \in Q_1^{cyc} \subseteq Q_0^{Aus}$. It is also similar to Case (2), only need note that α^+ corresponds to an injective morphism and β^- corresponds to a surjective morphism.

Therefore, $\langle I^{Aus} \rangle = \langle I^A \rangle$, and so $KQ^{Aus}/\langle I^{Aus} \rangle$ is isomorphic to the Cohen-Macaulay Auslander algebra of Λ. □

Corollary 3.6. *Let $\Lambda = KQ/\langle I \rangle$ be a gentle algebra. Then the Cohen-Macaulay Auslander algebra of Λ is also a gentle algebra.*

Proof. From the structure of Q^{Aus} and I^{Aus}, it is easy to see that $KQ^{Aus}/\langle I^{Aus} \rangle$ is a gentle algebra. □

4. Some representation properties of the Cohen-Macaulay Auslander agelbras for gentle algebras

Before going on, let us fix some notations. Let Λ be a gentle algebra and Γ be its Cohen-Macaulay Auslander algebra.

For any $M = ((M_i)_{i \in Q_0}, (M_\alpha : M_i \to M_j)_{(\alpha: i \to j) \in Q_1}) \in \operatorname{mod} \Lambda$, define a Γ-module $\widehat{M} = ((N_i, N_\alpha)_{i \in Q_0, \alpha \in Q_1^{cyc}}, (N_\beta)_{\beta \in Q_1^{Aus}})$ as follows:

• For any $i \in Q_0 \subseteq Q_0^{Aus}$, we set $N_i = M_i$; for any $\alpha \in Q_1^{cyc} \subseteq Q_0^{Aus}$, we set $N_\alpha = \operatorname{Im} M_\alpha$.

• For any arrow in Q_1^{Aus}, if it is of form $(\beta : i \to j) \in Q_1^{ncyc}$, then we set $N_\beta = M_\beta$; if it is of form $\beta^+ : i \to \beta$, or of form $\beta^- : \beta \to j$ for some $(\beta : i \to j) \in Q_1^{cyc}$, we set N_{β^+} and N_{β^-} to be the natural morphisms $(N_i = M_i) \to (\operatorname{Im} M_\beta = N_\beta)$ and $(N_\beta = \operatorname{Im} M_\beta) \to (M_j = N_j)$ respectively, which are induced by $M_\beta : M_i \to M_j$.

It is easy to see that $\widehat{M}$ is actually a Γ-module. Since Im is a functor, we can define a functor $\Phi : \operatorname{mod} \Lambda \to \operatorname{mod} \Gamma$ such that $\Phi(M) := \widehat{M}$, with the natural definition on morphisms.

Lemma 4.1 ([13]). *Keep the notations as above. Then Φ is a covariant additive functor from $\operatorname{mod} \Lambda$ to $\operatorname{mod} \Gamma$.*

Since (Q, I) is a subquiver of (Q^{Aus}, I^{Aus}), i.e. Λ is a subalgebra of Γ, we get a restriction functor $\operatorname{res} : \operatorname{mod} \Gamma \to \operatorname{mod} \Lambda$. Explicitly, for any $N = ((N_i, N_\alpha)_{i \in Q_0, \alpha \in Q_1^{cyc}}, (N_\beta)_{\beta \in Q_1^{Aus}}) \in \operatorname{mod} \Gamma$, $\operatorname{res}(N)$ is defined as follows:

• For any $i \in Q_0$, $(\operatorname{res}(N))_i = N_i$;

• For any arrow $(\alpha : i \to j) \in Q_1$, if $\alpha \in Q_1^{ncyc}$, we set $(\operatorname{res} N)_\alpha = N_\alpha$; if $\alpha \in Q_1^{cyc}$, we set $(\operatorname{res}(N))_\alpha = N_{\alpha^-} N_{\alpha^+}$.

Since Λ and Γ are gentle algebras, their indecomposable modules are either string modules or band modules. We describe the action of Φ and res on string modules as follows.

• For a string $w = \alpha_1^{\epsilon_1} \alpha_2^{\epsilon_2} \dots \alpha_n^{\epsilon_n} \in \mathcal{S}(\Lambda)$, denote its corresponding string module by $M(w)$. For $i = 1, \dots, n$, if $\alpha_i \in Q_1^{cyc}$, we replace α_i by $\alpha_i^- \alpha_i^+$, and get a word in Γ, which is denoted by $\iota(w)$. Then it is easy to see that $\iota(w) \in \mathcal{S}(\Gamma)$, we denote its string module by $N(\iota(w))$.

Note that
$$\underline{\dim} N(\iota(w)) = \underline{\dim} M(w) + \sum_{\substack{\alpha_i \in Q_1^{cyc}, \\ w=\alpha_1^{\epsilon_1}\alpha_2^{\epsilon_2}\dots\alpha_n^{\epsilon_n}}} \underline{\dim} S_{\alpha_i},$$
where S_{α_i} is the simple module corresponding to $\alpha_i \in Q_1^{cyc} \subseteq Q_0^{Aus}$. In this way, we get a map $\iota : \mathcal{S}(\Lambda) \to \mathcal{S}(\Gamma)$, which is injective. It is easy to see that $\Phi(M(w)) = N(\iota(w))$.

• For a string $v = \beta_1\beta_2 \dots \beta_n \in \mathcal{S}(\Gamma)$, denote its corresponding string module by $N(v)$. Obviously, $\mathrm{res}(N(v))$ is also a string module if $\mathrm{res}(N(v)) \neq 0$, we denote by $\pi^-(v)$ the string of $\mathrm{res}(N(v))$. Explicitly, we denote by v' the longest substring of v such that $s(v'), t(v') \in Q_0 \subseteq Q_0^{Aus}$, then $\pi^-(v)$ is constructed from v' by replacing $\alpha^-\alpha^+$ with α for each $\alpha \in Q_1^{cyc}$. Note that if $\mathrm{res}(N(v)) = 0$, then $\pi^-(v)$ is not defined. This only happens when $v = 1_{(\alpha,i)}$ for some $\alpha \in Q_1^{cyc} \subseteq Q_0^{Aus}$.

Besides, there exists the shortest string v'' with $s(v''), t(v'') \in Q_0 \subseteq Q_0^{Aus}$, such that v is a substring of v''. Then $\pi^+(v)$ is constructed from v'' by replacing $\alpha^-\alpha^+$ with α for each $\alpha \in Q_1^{cyc}$. Obviously, $\pi^+(v) \in \mathcal{S}(\Lambda)$, we denote its string module by $M(\pi^+(v))$.

In this way, we get two surjective maps $\pi^-, \pi^+ : \mathcal{S}(\Gamma) \to \mathcal{S}(\Lambda)$, in fact, $\pi^-\iota = \mathrm{Id} = \pi^+\iota$.

Example 4.2. *Keep the notations as in Example 2.3 (a) and Example 3.2 (a). Let $v = \alpha\gamma_2\beta$, which is a string in $\mathcal{S}(\Lambda)$. Then $\iota(v) = \alpha^{-1}\alpha^+\gamma_2\beta^+\beta^{-1}$, which is a string in $\mathcal{S}(\Gamma)$.*

For π^+ and π^-, we have $\pi^+(\alpha^+) = \alpha$, $\pi^+(\alpha^-) = \alpha$, and $\pi^-(\alpha^+) = 1_{(1,1)}$, $\pi^-(\alpha^-) = 1_{(2,1)}$. Let $w = \alpha^+\gamma_2\beta^+$, which is a string in $\mathcal{S}(\Gamma)$. Then $\pi^+(w) = \alpha\gamma_2\beta$, which is a string in $\mathcal{S}(\Lambda)$, and $\pi^-(w) = \gamma_2$, which is a string in $\mathcal{S}(\Lambda)$.

Note that $0 \leq l(c) - l(\iota\pi^-(c)) \leq 2$ for any string $c \in \mathcal{S}(\Gamma)$ such that $\pi^-(c)$ is defined.

Lemma 4.3. *Let $\Lambda = KQ/\langle I\rangle$ be a gentle algebra. Then Λ admits band modules if and only if the Cohen-Macaulay Auslander algebra* $\mathrm{Aus}(\mathrm{Gproj}\,\Lambda)$ *of Λ admits band modules.*

Proof. Let $b = \alpha_1\alpha_2 \cdots \alpha_{n-1}\alpha_n$ be a band in Λ. Then it is easy to see that $\iota(b)$ is also a band in $\mathrm{Aus}(\mathrm{Gproj}\,\Lambda)$.

Conversely, for any band c in $\mathrm{Aus}(\mathrm{Gproj}\,\Lambda)$, if $s(c) = t(c) \in Q_0 \subseteq Q_0^{Aus}$, it is easy to see that $\pi^-(c)$ is a band in Q. Otherwise, if $s(c) = t(c) \in Q_1^{cyc}$, then there exists $\alpha_1 \in Q_1^{cyc}$ such that $s(c) = \alpha_1 = t(c)$, which implies that c is of form $\alpha_1^+c_1\alpha_1^-$ or $(\alpha_1^-)^{-1}c_1(\alpha_1^+)^{-1}$, since there is only one arrow α_1^- starting from α_1 and one arrow α_1^+ ending to α_1. We only check it for the first form since the second is similar. Then $d = c_1\alpha_1^-\alpha_1^+$ is also a band in $\mathrm{Aus}(\mathrm{Gproj}\,\Lambda)$. Since $s(d) = t(d) = s(\alpha_1) \in Q_0$, from the definition of π^-, we get that $s(\pi^-(d)) = s(d) = t(d) = t(\pi^-(d))$. Together with $\pi^-(d^m) = (\pi^-(d))^m$ for any $m > 0$, it is easy to see that $\pi^-(d)$ is a band in Λ. □

Theorem 4.4. *Let $\Lambda = KQ/\langle I\rangle$ be a gentle algebra. Then Λ is representation-finite if and only if the Cohen-Macaulay Auslander algebra $\Gamma = \mathrm{Aus}(\mathrm{Gproj}\,\Lambda)$ of Λ is representation-finite.*

Proof. Theorem 3.5 shows that the Cohen-Macaulay Auslander algebra of Λ is $KQ^{Aus}/\langle I^{Aus}\rangle$.

If $\Gamma = \mathrm{Aus}(\mathrm{Gproj}\,\Lambda)$ is representation-finite, then there is no band in Γ. Lemma 4.3 yields that there is no band in Λ. For each string $w = \alpha_1\alpha_2 \cdots \alpha_n$ in $\mathcal{S}(\Lambda)$, we have $\iota(w) \in \mathcal{S}(\Gamma)$. Note that ι is injective. Since Γ is representation-finite and every string defines a unique string module, there are only finitely many strings in Γ, which implies that there are only finitely many strings in Λ. Since Λ admits no band module, we get that Λ is representation-finite.

Conversely, if Λ is representation-finite, then there is no band in Λ. Lemma 4.3 yields that there is no band in Γ. Let c be a string in $\mathcal{S}(\Lambda)$. For any string $v \in \mathcal{S}(\Gamma)$ such that $\pi^-(v) = c$, it is easy to see that $\iota(c)$ is a substring of v and v is of form $\iota(c)$, $\alpha\iota(c)$, $\iota(c)\beta$

or $\alpha\iota(c)\beta$ for some α, β or their inverses in $(Q_1^{cyc})^{\pm}$. Since $(Q_1^{cyc})^{\pm}$ is a finite set, there are only finitely many strings v in $\mathcal{S}(\Gamma)$ such that $\pi^-(v) = c$. Additionally, there are only finitely many strings in Λ, so there are only finitely many strings in Γ, and then $\Gamma = \mathrm{Aus}(\mathrm{Gproj}\,\Lambda)$ is representation-finite since Γ admits no band module. $\square$

For a gentle algebra $\Lambda = KQ/\langle I\rangle$, if any indecomposable Λ-module M is uniquely determined by its dimension vector, then there is no band module in Λ, since each band yields infinitely many indecomposable modules with the same dimension vector.

Lemma 4.5. *Let $\Lambda = KQ/\langle I\rangle$ be a gentle algebra such that there is no loop in Q. If any indecomposable Λ-module M is uniquely determined by its dimension vector, then for any arrow $\alpha \in Q_1$, there is no arrow from $t(\alpha)$ to $s(\alpha)$, i.e., there is no oriented 2-cycle in Q.*

Proof. If there is an arrow $\beta : t(\alpha) \to s(\alpha)$, then there are two strings $s(\alpha) \xrightarrow{\alpha} t(\alpha)$, $t(\alpha) \xrightarrow{\beta} s(\alpha)$. So there are two string modules with the same dimension vector, giving a contradiction. $\square$

Theorem 4.6. *Let $\Lambda = KQ/\langle I\rangle$ be a gentle algebra such that there is no loop in Q. If any indecomposable Λ-module M is uniquely determined by its dimension vector, then any indecomposable $\mathrm{Aus}(\mathrm{Gproj}\,\Lambda)$-module N is uniquely determined by its dimension vector.*

Proof. If any indecomposable Λ-module M is determined by its dimension vector, then there is no band in Λ and Lemma 4.3 yields that $\Gamma = \mathrm{Aus}(\mathrm{Gproj}\,\Lambda)$ admits no band. So there are only string modules in $\mathrm{mod}\,\Gamma$. We also get that any string in $\mathcal{S}(\Lambda)$ is uniquely determined by its dimension vector up to the equivalence relation ρ.

For any vector $v = ((v_i)_{i\in Q_0}, (v_\alpha)_{\alpha\in Q_1^{cyc}})$ which is a dimension vector of a string Γ-module, set v_1 to be $(v_i)_{i\in Q_0}$ and v_2 to be $(v_\alpha)_{\alpha\in Q_1^{cyc}}$. If there are two strings $c, d \in \mathcal{S}(\Gamma)$, such that $\underline{\dim}\, c = \underline{\dim}\, d = v$, then $l(c) = l(d)$. If $v_1 = 0$, then v is the dimension vector of a simple Γ-module S_α for some $\alpha \in Q_1^{cyc} \subseteq Q_0^{Aus}$, the result follows immediately since every simple module is uniquely determined by its dimension vector.

If $v_1 \neq 0$, then both $\pi^-(c)$ and $\pi^-(d)$ are well-defined, and v_1 is the dimension vector of the strings $\pi^-(c)$ and $\pi^-(d)$ in Λ. It follows that $\pi^-(c) \sim \pi^-(d)$ since $\underline{\dim}\,\pi^-(c) = \underline{\dim}\,\pi^-(d)$ and any string in Λ is uniquely determined by its dimension vector up to the equivalence relation ρ. After choosing suitable representatives, we can assume that $\pi^-(c) = \pi^-(d)$. We get that $\iota\pi^-(c) = \iota\pi^-(d)$ appears as substrings of c and d. Recall that $0 \le l(c) - l(\iota\pi^-(c)) \le 2$.

Case (1). If $l(c) = l(\iota\pi^-(c))$, then $c = \iota\pi^-(c)$, which also implies $d = \iota\pi^-(d)$ by $l(c) = l(d)$. Then $c = d$ since $\pi^-(c) = \pi^-(d)$ and ι is injective.

Case (2). $l(c) - l(\iota\pi^-(c)) = 1$. We assume that $\iota\pi^-(c) = \iota\pi^-(d)$ is

$$b_1 \overset{\alpha_1}{\text{——}} b_2 \overset{\alpha_2}{\text{——}} \cdots \overset{\alpha_{n-1}}{\text{——}} b_n \overset{\alpha_n}{\text{——}} b_{n+1}.$$

Suppose for a contradiction that c is not equivalent to d.

Since $\underline{\dim}\, c = \underline{\dim}\, d$, there exists some $\alpha \in Q_1^{cyc}$ such that c and d are of the following forms:

$$c_1: \quad \alpha \xleftarrow{\alpha^+} b_1 \overset{\alpha_1}{\text{——}} b_2 \overset{\alpha_2}{\text{——}} \cdots \overset{\alpha_{n-1}}{\text{——}} b_n \overset{\alpha_n}{\text{——}} b_{n+1},$$

$$c_2: \quad \alpha \xrightarrow{\alpha^-} b_1 \overset{\alpha_1}{\text{——}} b_2 \overset{\alpha_2}{\text{——}} \cdots \overset{\alpha_{n-1}}{\text{——}} b_n \overset{\alpha_n}{\text{——}} b_{n+1},$$

$$c_3: \quad b_1 \overset{\alpha_1}{\text{——}} b_2 \overset{\alpha_2}{\text{——}} \cdots \overset{\alpha_{n-1}}{\text{——}} b_n \overset{\alpha_n}{\text{——}} b_{n+1} \xrightarrow{\alpha^+} \alpha,$$

$$c_4: \quad b_1 \overset{\alpha_1}{\text{——}} b_2 \overset{\alpha_2}{\text{——}} \cdots \overset{\alpha_{n-1}}{\text{——}} b_n \overset{\alpha_n}{\text{——}} b_{n+1} \xleftarrow{\alpha^-} \alpha.$$

If $c = c_1$, then d can only be of form c_3 or c_4 since there is no loop in Q. First, if $d = c_3$, then $\pi^+(d) = \pi^+(\iota\pi^-(d))\alpha^{-1} = \pi^+(\iota\pi^-(c))\alpha^{-1}$, and $\pi^+(c) = \alpha\pi^+(\iota\pi^-(c))$. Then $\underline{\dim}\,\pi^+(d) = \underline{\dim}\,\pi^+(c)$, which means that $\pi^+(d) \sim \pi^+(c)$. If $\pi^+(d) = \pi^+(c)$, then from the definition of π^+, we get that $\alpha^- = \alpha_1$, $\alpha^+ = \alpha_2$, $\alpha_1 = \alpha_3$ and so on. So $\alpha_3 = \alpha^-$, which yields that $\alpha^-\alpha^+\alpha^-$ is a string. However, $t(\alpha) = t(\alpha^-) = s(\alpha^+) = s(\alpha)$, which means that α is a loop in Q, contradicts to the assumption of Q. If $\pi^+(d) = (\pi^+(c))^{-1}$, then $\pi^+(\iota\pi^-(c))\alpha^{-1} = (\alpha\pi^+(\iota\pi^-(c)))^{-1} = (\pi^+(\iota\pi^-(c)))^{-1}\alpha^{-1}$, which means that $\pi^+(\iota\pi^-(c)) = (\pi^+(\iota\pi^-(c)))^{-1}$, giving a contradiction to Remark 2.4.

Second, if $d = c_4$, then

$$\iota\pi^+(c): \quad b_{n+1} \xleftarrow{\alpha^-} \alpha \xleftarrow{\alpha^+} b_1 \xrightarrow{\alpha_1} b_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} b_n \xrightarrow{\alpha_n} b_{n+1}$$

is a string, and its starting point and ending point coincide. From $\iota\pi^+(d)$, it is easy to see that $(\iota\pi^+(c))^m$ is also a string for any $m > 0$, which implies that there is a band in Γ, giving a contradiction. In conclusion, $d = c$ if c is of form c_1.

For c is one of forms c_2, c_3 and c_4, the proof is similar to the above, we omit the proof here.

Case (3). $l(c) - l(\iota\pi^-(c)) = 2$. We assume that

$$\iota\pi^-(c) = \iota\pi^-(d): \qquad b_1 \xrightarrow{\alpha_1} b_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} b_n \xrightarrow{\alpha_n} b_{n+1}.$$

There are four cases for the structure of c.

Case (3a). c is

$$c: \quad \alpha \xleftarrow{\alpha^+} b_1 \xrightarrow{\alpha_1} b_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} b_n \xrightarrow{\alpha_n} b_{n+1} \xrightarrow{\beta^+} \beta$$

for some $\alpha, \beta \in Q_1^{cyc}$. If $\alpha = \beta$, then $d = c$ since $\underline{\dim}\, c = \underline{\dim}\, d$ and Q has no loop.

For $\alpha \neq \beta$, suppose for a contradiction that d is not equivalent to c. Then d is one of the following forms:

$$d_1: \quad \beta \xleftarrow{\beta^+} b_1 \xrightarrow{\alpha_1} b_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} b_n \xrightarrow{\alpha_n} b_{n+1} \xrightarrow{\alpha^+} \alpha,$$

$$d_2: \quad \beta \xleftarrow{\beta^+} b_1 \xrightarrow{\alpha_1} b_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} b_n \xrightarrow{\alpha_n} b_{n+1} \xleftarrow{\alpha^-} \alpha,$$

$$d_3: \quad \beta \xrightarrow{\beta^-} b_1 \xrightarrow{\alpha_1} b_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} b_n \xrightarrow{\alpha_n} b_{n+1} \xrightarrow{\alpha^+} \alpha,$$

$$d_4: \quad \beta \xrightarrow{\beta^-} b_1 \xrightarrow{\alpha_1} b_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} b_n \xrightarrow{\alpha_n} b_{n+1} \xleftarrow{\alpha^-} \alpha.$$

For $d = d_1$, if $n = 0$, then $d = c^{-1}$, a contradiction. If $n > 0$, then there are two arrows α^+, β^+ from b_1, and α_1 is of form $\alpha_1 : b_2 \to b_1$ since Γ is gentle. Then $\beta^+\alpha_1, \alpha^+\alpha_1 \notin I^{Aus}$, a contradiction. For $d = d_2$, if $n = 0$, then there is an oriented 2-cycle $b_1 \xrightarrow{\alpha^+} \alpha \xrightarrow{\alpha^-} b_1$ in Γ, a contradiction; if $n > 0$, then similar to the above case $d = d_1$, we can get that it is also impossible. For $d = d_3$, it is easy to see that $b_{n+1} = b_1$, then there is an oriented 2-cycle $b_1 \xrightarrow{\beta^+} \beta \xrightarrow{\beta^-} b_1$, a contradiction. For $d = d_4$, there is an oriented 2-cycle $b_{n+1} \xrightarrow{\beta} b_1 \xrightarrow{\alpha} b_{n+1}$ in Q, a contradiction to Lemma 4.5. Therefore, d is equivalent to c in this case.

Case (3b). c is of form

$$c: \quad \alpha \xleftarrow{\alpha^+} b_1 \xrightarrow{\alpha_1} b_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} b_n \xrightarrow{\alpha_n} b_{n+1} \xleftarrow{\beta^-} \beta$$

for some $\alpha, \beta \in Q_1^{cyc}$. If $\alpha = \beta$, then $d = c$ since $\underline{\dim}\, c = \underline{\dim}\, d$ and Q has no loop. For $\alpha \neq \beta$, suppose for a contradiction that d is not equivalent to c. Then d is also one of the forms d_1, d_2, d_3, d_4 as described in Case (3a).

For $d = d_1$, if $n = 0$, then $b_1 \xrightarrow{\beta^+} \beta \xrightarrow{\beta^-} b_1$ is an oriented 2-cycle, a contradiction. If $n > 0$, then we can check that it is impossible similar to Case (3a). For $d = d_2$, if $n = 0$, then $b_1 \xrightarrow{\beta^+} \beta \xrightarrow{\beta^-} b_1$ is an oriented 2-cycle, a contradiction. If $n > 0$, then there are two arrows α^-, β^- ending to b_{n+1}, and α_n is of form $\alpha_n : b_{n+1} \to b_n$ since Γ is gentle. Then $\alpha_n\beta^-, \alpha_n\alpha^- \notin I^{Aus}$, a contradiction. For $d = d_3$, it is easy to see that $\underline{\dim}\, \pi^+(d) = \underline{\dim}\, \pi^+(c)$, so $\pi^+(d) \sim \pi^+(c)$ and then $\iota\pi^+(d) \sim \iota\pi^+(c)$, that is

$$\iota\pi^+(c): \quad t(\alpha) \xleftarrow{\alpha^-} \alpha \xleftarrow{\alpha^+} b_1 \overset{\alpha_1}{\rule[0.5ex]{2em}{0.4pt}} b_2 \overset{\alpha_2}{\rule[0.5ex]{2em}{0.4pt}} \cdots \overset{\alpha_{n-1}}{\rule[0.5ex]{2em}{0.4pt}} b_n \overset{\alpha_n}{\rule[0.5ex]{2em}{0.4pt}} b_{n+1} \xleftarrow{\beta^-} \beta \xleftarrow{\beta^+} s(\beta)$$

and

$$\iota\pi^+(d): \quad s(\beta) \xrightarrow{\beta^+} \beta \xrightarrow{\beta^-} b_1 \overset{\alpha_1}{\rule[0.5ex]{2em}{0.4pt}} b_2 \overset{\alpha_2}{\rule[0.5ex]{2em}{0.4pt}} \cdots \overset{\alpha_{n-1}}{\rule[0.5ex]{2em}{0.4pt}} b_n \overset{\alpha_n}{\rule[0.5ex]{2em}{0.4pt}} b_{n+1} \xrightarrow{\alpha^+} \alpha \xrightarrow{\alpha^-} t(\alpha)$$

are equivalent under ρ, which implies that $\iota\pi^+(c) = (\iota\pi^+(d))^{-1}$. Then $(\iota\pi^-(c)) = (\iota\pi^-(c))^{-1}$, which is impossible. For $d = d_4$, obviously, $b_{n+1} = b_1$ and so $b_1 \xrightarrow{\alpha^+} \alpha \xrightarrow{\alpha^-} b_1$ is an oriented 2-cycle, a contradiction. Therefore, in this case, d is equivalent to c.

Case (3c). c is of form

$$c: \quad \alpha \xrightarrow{\alpha^-} b_1 \overset{\alpha_1}{\rule[0.5ex]{2em}{0.4pt}} b_2 \overset{\alpha_2}{\rule[0.5ex]{2em}{0.4pt}} \cdots \overset{\alpha_{n-1}}{\rule[0.5ex]{2em}{0.4pt}} b_n \overset{\alpha_n}{\rule[0.5ex]{2em}{0.4pt}} b_{n+1} \xrightarrow{\beta^+} \beta$$

for some $\alpha, \beta \in Q_1^{cyc}$. This case is similar to Case (3b), we omit the proof here.

Case (3d). c is

$$c: \quad \alpha \xrightarrow{\alpha^-} b_1 \overset{\alpha_1}{\rule[0.5ex]{2em}{0.4pt}} b_2 \overset{\alpha_2}{\rule[0.5ex]{2em}{0.4pt}} \cdots \overset{\alpha_{n-1}}{\rule[0.5ex]{2em}{0.4pt}} b_n \overset{\alpha_n}{\rule[0.5ex]{2em}{0.4pt}} b_{n+1} \xleftarrow{\beta^-} \beta$$

for some $\alpha, \beta \in Q_1^{cyc}$. This case is similar to Case (3a), we omit the proof here.

To sum up, when $l(c) - l(\iota\pi^-(c)) = 2$, we get that c is equivalent to d.

Therefore, for any strings c, d in $\mathcal{S}(\Gamma)$, if $\underline{\dim}\, c = \underline{\dim}\, d$, then $c \sim d$. For any indecomposable Γ-module N, we get that N is a string module, which is uniquely determined by its string up to the equivalent relation ρ, and so N is uniquely determined by its dimension vector. $\square$

The following example shows that the converse of Theorem 4.6 is not valid.

Example 4.7. *Let $\Lambda = KQ/\langle I\rangle$ be a gentle algebra with*

$$Q: \qquad 1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2 \qquad\qquad I = \{\alpha\beta, \beta\alpha\}.$$

Then Q^{Aus} is as following diagram shows and $I^{Aus} = \{\beta^+\alpha^-, \alpha^+\beta^-\}$.

$$\begin{tikzcd}
& \alpha \arrow[dr, "\alpha^-"] & \\
1 \arrow[ur, "\alpha^+"] & & 2 \arrow[dl, "\beta^+"] \\
& \beta \arrow[ul, "\beta^-"] &
\end{tikzcd}$$

It is easy to see that any indecomposable $KQ^{Aus}/\langle I^{Aus}\rangle$-module is uniquely determined by its dimension vector. However, the indecomposable projective Λ-modules P_1, P_2 corresponding to vertices $1, 2$ respectively, have the same dimension vector.

Remark 4.8. *Let $\Lambda = KQ/\langle I\rangle$ be a gentle algebra. If any indecomposable Λ-module M is uniquely determined by its dimension vector, then for any loop $\alpha : i \to i$ with i a vertex, there is no arrow $\beta \neq \alpha$ starting from i or ending to i.*

Proof. Since Λ is a gentle algebra, for any loop $\alpha : i \to i$, we have $\alpha^2 \in I$. First, note that there is not another loop β with the same starting point i. Otherwise, we also have $\beta^2 \in I$. Then $\beta\alpha, \alpha\beta \notin I$ since Λ is gentle, contradicts to the fact Λ is finite-dimensional.

If there is another arrow $\beta : i \to j$, then $j \neq i$. Obviously, $\beta\alpha \notin I$. So there are two nonequivalent strings $i \xrightarrow{\alpha} i \xrightarrow{\beta} j$ and $i \xleftarrow{\alpha} i \xrightarrow{\beta} j$, which have the same dimension vector, a contradiction.

If there is another arrow $\beta : j \to i$, it is similar to the above case, we omit the proof here. $\square$

Example 4.9. *Let $\Lambda = KQ/\langle I\rangle$ be a gentle algebra with $Q_0 = \{1\}$, $Q_1 = \{\alpha : 1 \to 1\}$. Then $I = \{\alpha^2\}$. Let $KQ^{Aus}/\langle I^{Aus}\rangle$ be the Cohen-Macaulay Auslander algebra of Λ. Then Q^{Aus} is as the following diagram shows and $I^{Aus} = \{\alpha^+\alpha^-\}$.*

$$Q^{Aus}: \qquad 1 \underset{\alpha^-}{\overset{\alpha^+}{\rightleftarrows}} 2$$

It is easy to that $KQ^{Aus}/\langle I^{Aus}\rangle$ does not satisfy that any indecomposable module is uniquely determined by its dimension vector.

Corollary 4.10. *Let $\Lambda = KQ/\langle I\rangle$ be a gentle algebra with Q connected. Assume that Λ satisfies that any indecomposable Λ-module M is uniquely determined by its dimension vector. If there are two indecomposable $\mathrm{Aus}(\mathrm{Gproj}(\Lambda))$-modules with the same dimension vector, then Λ is isomorphic to the local ring $K[X]/\langle X^2\rangle$.*

Proof. Since any indecomposable Λ-module M is uniquely determined by its dimension vector, if there is no loop in Q, Theorem 4.6 yields that any indecomposable $\mathrm{Aus}(\mathrm{Gproj}\,\Lambda)$-module N is determined by its dimension vector, a contradiction. So there is at least one loop in Q. Furthermore, Remark 4.8 implies that $Q_0 = \{v\}$, $Q_1 = \{\alpha : v \to v\}$ since Q is connected, and so $\Lambda \cong K[X]/\langle X^2\rangle$. $\square$

At the end of this section, we give the following proposition for schurian gentle algebras. Recall that an algebra $A = KQ/I$ is *schurian* if $\dim_k \mathrm{Hom}_A(P_i, P_j) \leq 1$ for any two vertices i, j of Q, or in other words, the entries of its Cartan matrix are only 0 or 1.

Proposition 4.11. *Let $\Lambda = KQ/\langle I\rangle$ be a schurian gentle algebra. Then its Cohen-Macaulay Auslander algebra $\Gamma = \mathrm{Aus}(\mathrm{Gproj}\,\Lambda)$ is also a schurian gentle algebra.*

Proof. Let P be an indecomposable projective Γ-module corresponding to some vertex $b_1 \in Q_0^{Aus}$. Since Γ is a gentle algebra, P is a string module, see e.g. [17, Section 4]. Denote by w its string. Then from [17, Section 4], we get that w is of form

$$w: \quad b_{n+m+1} \xleftarrow{\beta_m} \cdots \xleftarrow{\beta_2} b_{n+2} \xleftarrow{\beta_1} b_1 \xrightarrow{\alpha_1} b_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} b_n \xrightarrow{\alpha_n} b_{n+1},$$

or

$$w: \quad a_1 \xrightarrow{\gamma_1} a_2 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_{l-1}} a_l \xrightarrow{\gamma_l} a_{l+1},$$

where the paths $\beta_m \dots \beta_2\beta_1$, $\alpha_n \dots \alpha_2\alpha_1$ and $\gamma_l \dots \gamma_2\gamma_1$ appearing above are maximal, e.g. there does not exist $\beta \in Q_1^{Aus}$ such that $\beta\beta_m \notin I^{Aus}$, see e.g. [2, 17]. Therefore, we only need to check that the string w passes through any vertex at most once.

For w is of the first case, we claim that $b_1, b_{n+1}, b_{n+m+1} \in Q_0 \subseteq Q_0^{Aus}$. In fact, if $b_1 \notin Q_0$, then $b_1 = \alpha$ for some $\alpha \in Q_1^{cyc} \subseteq Q_0^{Aus}$. Then there are two arrows α_1, β_1 starting from α. Recall that there is only one arrow α^- starting from α in Q^{Aus}, a contradiction. If $b_{n+1} \notin Q_0$, then $b_{n+1} = \beta$ for some $\beta \in Q_1^{cyc} \subseteq Q_0^{Aus}$. Since there is only one arrow β^+ ending to β in $\mathbb{Q}^{Aus}$, $\alpha_n = \beta^+$. However, $\beta^-\beta^+ \notin I^{Aus}$, so we get that $\alpha_n \cdots \alpha_2\alpha_1$ is not maximal, a contradiction. For $b_{n+m+1} \in Q_0$, it is similar to the above.

It is easy to see that $\pi^-(w) \in \mathcal{S}(\Lambda)$ is the string of the indecomposable projective Λ-module corresponding to the vertex $b_1 \in Q_0$. From Λ is schurian, we get that $\pi^-(w)$ does not pass through any vertex more than once. It follows that w does not pass through any vertex in $Q_0 \subseteq Q_0^{Aus}$ more than once. Furthermore, if w passes through a vertex $\alpha \in Q_1^{cyc} \subseteq Q_0^{Aus}$ at least twice, then w must pass through $s(\alpha)$ or $t(\alpha)$ at least twice, which yields that $\pi^-(w)$ passes through $s(\alpha)$ or $t(\alpha)$ at least twice, a contradiction.

If w is of the second case, similar to the first case, we get that $a_{l+1} \in Q_0 \subseteq Q_0^{Aus}$. If $a_1 \in Q_0$, then it is similar to the first case. If $a_1 = \alpha \in Q_1^{cyc}$, then $\alpha_1 = \alpha^-$ since there is only one arrow α^- starting from α. It is easy to see that $\pi^+(w) \in \mathcal{S}(\Lambda)$ is the string of a quotient of the indecomposable projective Λ-module ${}_\Lambda P_{s(\alpha)}$ corresponding to the vertex $s(\alpha)$. Let v be the string of ${}_\Lambda P_{s(\alpha)}$. From the above, we know that v does not pass through any vertex more than once. Note that w is a substring of v, so w does not pass through any vertex more that once.

Therefore, Γ is a schurian algebra. $\square$

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